SYNTHESIS OF DOUBLY SYMMETRIC VIBRATING BEAMS INCLUDING SHEAR

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THESIS

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June 1969

Synthesis of Doubly Symmetric
Vibrating Beams Including Shear

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Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN MECHANICAL ENGINEERING

from the

NAVAL POSTGRADUATE SCHOOL
June 1969

ABSTRACT

A fundamental, non-linear, ordinary differential equation is derived for the flexural vibrations of a non-uniform beam, including shear.

This equation is solved with the specializing assumptions of a constant shear coefficient and that the moment of inertia is directly proportional to the cross sectional area.

A perturbation method is used and the eigen frequencies and mode shapes obtained are functions of a perturbation parameter ϵ. These solutions are accurate up to the second order of ϵ.

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SYMBOLS AND UNITS

ENGLISH LETTER SYMBOLS

A	Beam cross sectional area, L^2
A*, B*	Lengthy combinations of other quantities (see equations (14) and (14a)
C_1	Proportionality constant, $I = C_1 A$, L^2
E	Young's modulus, FL^{-2}
G	Shear modulus, FL^{-2}
I	Moment of inertia, L^4
k	Shear coefficient, dimensionless
L	Beam length, L
M	Bending moment, FL
s	$kAG(y'-\psi)$, F
t	Time, T
V	Shearing force, F
x	Distance along beam, L
y	Beam deflection (positive downward), L

GREEK LETTER SYMBOLS

α	$\frac{\pi x}{L}$, radians
θ	$\frac{EC_1}{kG}$, L
ρ	Mass density, $FT^2 L^{-4}$
σ	$\frac{1}{kAG}$, F^{-1}
ψ	Slope of beam centerline due to bending
ω	Frequency of vibration, T^{-1}

OTHER SYMBOLS

$'$	$\frac{\partial}{\partial x}$
$\cdot$	$\frac{\partial}{\partial t}$

Symbols not seen here may be found in Appendix A.

ACKNOWLEDGEMENT

Thanks are expressed to Eugene F. Lynch, Assistant Professor of Mechanical Engineering, Naval Postgraduate School, whose advice and counsel made this thesis possible.

INTRODUCTION

The Timoshenko beam equations have been well established [5]*. Neglecting rotary inertia, these equations become:

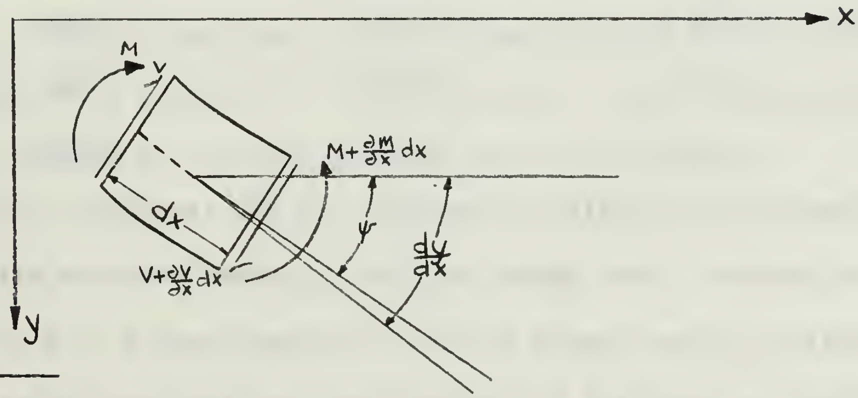
$$\frac{d}{dx}\left[EI\frac{d\psi}{dx}\right] + kAG\left(\frac{dy}{dx}-\psi\right)=0 \tag{1}$$

$$\rho A\frac{d^2y}{dt^2} - \frac{d}{dx}\left[kAG\left(\frac{dy}{dx}-\psi\right)\right]=0 \tag{2}$$

The left hand sides of both equations (1) and (2) are functions of y and ψ. Because of this common dependence, these equations are referred to as coupled. Uncoupling of equations (1) and (2) involves their reduction to one equation in which either y or ψ, but not both, appear.

It has been shown that the correction for shear is much greater than that for rotary inertia [10]. The effect of shear becomes increasingly important as the frequency of vibration is increased [9]. Therefore, the effects of shear are worth investigating. Figure (1) is a free body diagram of a beam element. Note that the quantity $(\frac{dy}{dx}-\psi)$ in Figure (1) is the slope of the beam centerline due to shear deformation.

Figure 1



*List of references appears on page 53

The shear term in equation (1) contains the shear coefficient, k. Timoshenko defined the shear coefficient as average shear stress over maximum shear stress. This shear coefficient, then, is a function of the cross sectional area of the beam. Timoshenko found it to be constant for general classes of geometrical shapes. However, there have been other studies investigating this shear coefficient which have disagreed with Timoshenko's values. In 1966, G. R. Cowper [4] derived expressions for the shear coefficient which were functions of the geometry, Poisson's ratio, and the dimensions of the cross section in some instances. R. D. Mindlin and H. Deresiewicz [9] maintain that Timoshenko's shear coefficient is a function also of the frequency of vibration. The following table contains some representative values for the shear coefficient. These values were determined by different methods and different assumptions. Therefore their comparison is difficult.

Table 1 (From Cowper [4])

		k for rectangle	k for circle
Timoshenko		0.667	0.750
Mindlin		0.822	0.847
Cowper	$\upsilon = 0$	0.833	0.857
	$\upsilon = .3$	0.850	0.886
	$\upsilon = .5$	0.870	0.900

There seems to be a conflict of opinions in the literature on the shear coefficient. This paper develops a general equation with k as a function of beam length but all solutions treat k as a constant. This is justified by the fact that for the class of areas investigated, that is, where the moment of inertia is directly proportional to the cross sectional area, there is general agreement that k is a constant.

The remainder of this paper will be devoted to the uncoupling of equations (1) and (2) and the subsequent solving of the uncoupled equation for its eigen frequencies and associated mode shapes. This uncoupled equation will be general with respect to cross sectional area, moment of inertia, and shear coefficient. That is, these quantities will be functions of the distance along the beam. Solutions of this type would be of use in optimization of design. It is often desirable to minimize deflection, bending stress or modify frequency of vibration in beam design. By allowing the cross sectional area of the beam to vary with x it may be possible to reduce deflection and bending stress or to modify the frequency of vibration through appropriate selection of an area function. In this thesis the area function will be defined in such a way that only one dimension of the cross sectional area be a function of x. For the sake of comparison to existing solutions, this variable dimension must vary in such a way as to keep the beam volume constant. That is, the beam volume must be the same if the variable dimension is a constant or a function of x. This stipulation will become evident in the derivation of the area function.

DEVELOPMENT OF THE GOVERNING EQUATION

With the substitutions:

$$S = kAG(y' - \psi) \tag{5}$$

$$\sigma = \frac{1}{kAG} \tag{6}$$

equations (1) and (2) become:

$$[EI\psi']' + S = 0 \tag{7}$$

$$\rho A\ddot{y} = S' \tag{8}$$

where $' = \frac{\partial}{\partial x}$ and $\dot{} = \frac{\partial}{\partial t}$

Solving equation (5) for ψ:

$$\psi = y' - S\sigma \tag{9}$$

$$\psi' = y'' - (S\sigma)' \tag{10}$$

$$\psi'' = y''' - (S\sigma)'' \tag{11}$$

Equation (7) is:

$$EI\psi'' + EI'\psi' + S = 0 \tag{12}$$

Substitution of equation (10) and (11) into (12) gives:

$$EI[y''' - S''\sigma - 2S'\sigma'] + EI'[y'' - S'\sigma] + S[1 - (EI\sigma')'] = 0 \tag{13}$$

Substituting equation (8) into (13):

$$EI[y''' - \sigma(\rho A\ddot{y})' - 2\sigma'\rho A\ddot{y}] + EI'[y'' - \sigma\rho A\ddot{y}] + S[1 - (EI\sigma')'] = 0 \tag{14}$$

Writing equation (14) as:

$$A^* + S B^* = 0 \tag{14a}$$

thus,

$$S = -\frac{A^*}{B^*} \tag{15}$$

where $B^* \neq 0$ (see Appendix B)

Differentiating equation (15) with respect to x gives:

$$\rho A \ddot{y} = \frac{A^* B^{*\prime} - A^{*\prime} B^*}{B^{*2}}$$

$$\rho A \ddot{y} B^{*2} + A^{*\prime} B^* - A^* B^{*\prime} = 0 \tag{16}$$

This form of the equation does not seem to have been previously reported. It is clear that this single equation is fundamental to the study of vibrations of non-uniform beams, including shear.

For the case of a prismatic beam equation (16) reduces to:

$$EI \frac{\partial^4 y}{\partial x^4} - \frac{EI\rho}{kG} \frac{\partial^4 y}{\partial x^2 \partial t^2} + \rho A \frac{\partial^2 y}{\partial t^2} = 0 \tag{16a}$$

From this point on, the analysis in this thesis will focus upon and be confined to the case of simply supported beams for which the end conditions are:

$$(y)_{x=0} = (y)_{x=L} = (y'')_{x=0} = (y'')_{x=L} = 0 \tag{17}$$

These conditions prescribe zero deflection and zero moment at both ends of the beam. Solutions to equation (16a) are:

$$y_n = g_n(x) \sin(\omega_n t + \phi) \quad ; \quad n = 1, 2, 3 \ldots \tag{18}$$

$$g_n(x) = \sin \lambda_n x \quad ; \quad \lambda_n = \frac{n\pi}{L} \tag{19}$$

where:

$$\omega_n^2 = \frac{EI \lambda_n^4}{\left[\frac{EI}{kG}\rho \lambda_n^2 + \rho A\right]}$$

are the eigen frequencies of vibration for the uniform beam. The associated mode shapes are the corresponding g_n's.

Equation (16) is completely general and it is clear that difficulty will attend attempts to find useful solutions which also satisfy equation (17). Therefore, in order to make the equation more tractable, a specialization is made at this point. The area function is limited to widthwise variations so that the moment of inertia is directly proportional to the cross sectional area and the shape of the cross section is limited to those for which the shear coefficient is constant.

After expansion and combination, equation (16) becomes:

$$
\begin{aligned}
&A\left[EC_1 y^{IV} - \rho\omega^2 y + \Theta\rho\omega^2 y''\right] + A'\left[2EC_1 y'''\right] - \\
&\left\{\frac{2A''A'^2 - AA''^2}{A^2}\right\}\left[\Theta^2\rho\omega^2 y + \Theta EC_1 y''\right] - \frac{A'A''}{A}\left[\Theta^2\rho\omega^2 y' \right. \\
&\left. + \Theta EC_1 y'''\right] + \frac{A'^2}{A}\left[\Theta^2\rho\omega^2 y'' + \Theta EC_1 y^{IV} - \Theta\rho\omega^2 y\right] + \\
&\left\{\frac{2A'^4 - A'^2A''A}{A^3}\right\}\left[2\Theta EC_1 y'' + 5\Theta^2\rho\omega^2 y\right] + \\
&\left\{\frac{6AA'^2A'' - A^2A'A''' - 6A'^4}{A^3}\right\}\left[\Theta EC_1 y'' - \Theta^2\rho\omega^2 y\right] + \\
&\frac{A'^3}{A^2}\left[\Theta EC_1 y''' + 3\Theta^2\rho\omega^2 y'\right] - \left\{\frac{2A'^3 - AA'A''}{A^2}\right\}\left[\Theta EC_1 y''' \right. \\
&\left. + 4\Theta^2\rho\omega^2 y'\right] + \left\{\frac{6AA'A'' - A^2A''' - 6A'^3}{A^2}\right\}\left[\Theta EC_1 y''' + \right. \\
&\left. \Theta^2\rho\omega^2 y'\right] - \left\{\frac{2A'^2 - AA''}{A}\right\}\left[\Theta EC_1 y^{IV} - 4\Theta\rho\omega^2 y + \Theta^2\rho\omega^2 y''\right] \\
&+ A''\left[EC_1 y'' + \Theta\rho\omega^2 y\right] - \frac{A'^3}{A^2}\left[3\Theta^2\rho\omega^2 y'\right] - \left\{\frac{4A'^4 + A^2A''^2 - 4A'^2A''A}{A^3}\right\} \\
&\left[3\Theta^2\rho\omega^2 y\right] + \frac{A'^4}{A^3}\left[4\Theta^2\rho\omega^2 y\right] + \frac{A'^2A''}{A^2} \\
&\left[2\Theta^2\rho\omega^2 y\right] = 0
\end{aligned}
\tag{20}
$$

Equation (20) is the governing differential equation for the vibrating Timoshenko beam, neglecting rotary inertia, when the shear coefficient is constant and the moment of inertia is directly proportional to the cross sectional area.

SOLUTION OF AREA FUNCTION

Because of the high degree of non-linearity of equation (20), it does not lend itself to closed form solution. Accordingly, an approximate method of solution is employed. It is desirable to obtain a solution which permits verification for a prismatic beam. With these ends in mind, a perturbation expansion of the symmetric mode shapes is considered.

$$y = \sin\frac{\pi x}{L} + \epsilon \sin 3\frac{\pi x}{L} + \epsilon^2 \sin 5\frac{\pi x}{L} \tag{21}$$

Physical arguments suggest that the substitution of equation (21) into (20) will generate a symmetric area function. Substitution of equation (21) into (20) yields a differential equation with the parameter ϵ in it, and we will take ϵ to be small. That is, a perturbation method of solution will be employed. This solution will be a perturbation from the fundamental mode of the prismatic beam solution. If the variable area function is appropriately defined, the perturbation method will provide a link to the prismatic beam solution. This link will be effected by setting the perturbation parameter equal to zero in the expression for the variable area. Let:

$$A = A_0 + \epsilon A_1 + \epsilon^2 A_2 \tag{22}$$

$$\omega^2 = \omega_0^2 + \epsilon\,\omega_1^2 + \epsilon^2 \omega_2^2 \tag{23}$$

In equation (22) A_0 is presumed to be constant since the solution we are seeking is a perturbation with respect to the solution for a prismatic beam.

It is expected that the solutions of equation (20) will be valid only for small values of ϵ since they will be perturbations from the prismatic beam solutions.

When the perturbation expansions (21), (22) and (23) are substituted into equation (20), area terms of higher order than ϵ^2 are generated. These terms may be neglected because the perturbation, in this case, considers only terms of second or lower order. For example, consider the third term of equation (20):

$$A'^2 A'' = [\epsilon A_1' + \epsilon^2 A_2']^2 [\epsilon A_1'' + \epsilon^2 A_2'']$$
$$= [\epsilon^2 A_1'^2 + \epsilon^4 A_2'^2 + 2\epsilon^3 A_1' A_2'][\epsilon A_1'' + \epsilon^2 A_2'']$$

The lowest order of ϵ generated here is ϵ^3, therefore, all terms are of higher order than ϵ^2 and the term $A'^2 A'$ may be neglected.

When equations (21), (22) and (23) are substituted into equation (20) and common powers of ϵ are equated, three linear differential equations in A_0, A_1, and A_2 are generated. The equation for A_0 is simply the constant cross sectional area case.

$$EA_0C_1y_0^{IV} + \Theta A_0 \rho \omega_0^2 y_0'' - \rho A_0 \omega_0^2 y_0 = 0$$

For a simply supported beam:

$$\omega_0^2 = \frac{EC_1(\frac{\pi}{L})^4}{\rho + \Theta\rho(\frac{\pi}{L})^2}$$

ω_0^2 is the leading frequency term in the perturbation expansion (23).

The first order of ϵ generates a linear differential equation in $A_1(x)$.

$$\begin{aligned}
&A_1'''[-\Theta^2\rho\omega_0^2 y_0' - \Theta EC_1 y_0'''] - A_1''[3\Theta\rho\omega_0^2 y_0 - \Theta^2\rho\omega_0^2 y_0'' \\
&- EC_1 y_0'' - \Theta EC_1 y_0^{IV}] + A_1'[2EC_1 y_0'''] + A_1[2EC_1 y_0^{IV} \\
&- 2\rho\omega_0^2 y_0 + 2\Theta\rho\omega_0^2 y_0''] = -A_0[EC_1 y_1^{IV} - \rho\omega_0^2 y_1 \\
&- \rho\omega_1^2 y_0 + \Theta\rho\omega_0^2 y_1'' + \Theta\rho\omega_1^2 y_0'']
\end{aligned} \tag{24}$$

This is a linear differential equation with known variable coefficients.

The only restriction on the total area function is that the volume of the beam must remain constant. Therefore:

$$\int_A A\,dx = \text{Volume} = A_0 L + \epsilon \int_A A_1\,dx + \epsilon^2 \int_A A_2\,dx \tag{25}$$

The perturbation parameter ϵ is arbitrary thus defining a class of areas. Therefore, to keep the beam volume constant for all ϵ, it is sufficient to require that:

$$\int_A A_1\,dx = \int_A A_2\,dx = 0 \tag{26}$$

Assuming that $A_1(x)$ is piecewise regular and symmetric it may be represented by a Fourier series of the form:

$$A_1 = \sum_{i=0}^{\infty} \left[N_i \sin \frac{i\pi x}{L} + M_i \cos \frac{i\pi x}{L} \right]$$

Equation (26) restricts the arguments of the Fourier expansion to even powers of $\frac{\pi x}{L}$.

$$\int_0^L A_1\,dx = \int_0^L \sum_{i=0}^{\infty} \left[N_i \sin \frac{i\pi x}{L} + M_i \cos \frac{i\pi x}{L} \right] dx = 0$$

$$= \sum_{i=0}^{\infty} \left[-\frac{L}{i\pi} N_i (\cos i\pi - 1) + \frac{L}{i\pi} M_i \sin i\pi \right] = 0$$

$$\sin i\pi = 0 \quad ; \; i = 0, 1, 2, 3, \ldots$$

$$\cos i\pi - 1 = 0 \; ; \; i = 0, 2, 4, \ldots$$

Therefore, for $\int_0^L A_1\,dx = 0$, it is sufficient that the arguments of the sine and the cosine functions in the Fourier series be even.

Substituting:

$$y_0 = \sin \frac{\pi x}{L}$$

$$y_1 = \sin \frac{3\pi x}{L} \quad , \; \alpha = \frac{\pi x}{L}$$

into equation (11) gives:

$$A_1''' K_1 \cos\alpha + A_1'' K_2 \sin\alpha - A_1' K_3 \cos\alpha - A_1 K_4 \sin\alpha = A_0 \left[K_5 \sin 3\alpha + K_6 \sin\alpha \right] \tag{27}$$

Examination of equation (27) shows that only the cosine terms in the Fourier expansion remain in the solution for $A_1(x)$. If the sine terms were allowed to remain they would generate products of cos α and cos $n\alpha$ and products of sin α and sin $n\alpha$. These products can only be represented as cosine functions. The right-hand side of equation (27) contains only sine terms. Therefore the coefficients of the sine terms in the Fourier expansion must all be zero. On the other hand the cosine terms in the Fourier expansion generate only products of sine and cosine which may be represented entirely as sine functions through trigonometric identities. Therefore:

$$A_1(x) = \sum_{n=0}^{\infty} M_n \cos n\alpha$$

Substitution of this expansion into equation (27) gives:

$$\sum_{n=0}^{\infty} M_n\Big[\left(\tfrac{n\pi}{L}\right)^3 K_1\left\{\tfrac{1}{2}\sin(n\alpha+\alpha)+\tfrac{1}{2}\sin(n\alpha-\alpha)\right\} - \left(\tfrac{n\pi}{L}\right)^2 K_2\left\{\tfrac{1}{2}\sin(n\alpha+\alpha)\right.$$
$$\left.+\tfrac{1}{2}\sin(\alpha-n\alpha)\right\}+\left(\tfrac{n\pi}{L}\right)K_3\left\{\tfrac{1}{2}\sin(n\alpha+\alpha)+\tfrac{1}{2}\sin(n\alpha-\alpha)\right\} - \qquad (28)$$
$$K_4\left\{\tfrac{1}{2}\sin(n\alpha+\alpha)+\tfrac{1}{2}\sin(\alpha-n\alpha)\right\}\Big] = A_0\left[K_5\sin 3\alpha + K_6 \sin\alpha\right]$$

There are only two values of n which can satisfy this equation:

$$n = \pm 2$$

n is taken equal to + 2 because,

$$\cos 2\alpha = \cos(-2\alpha)$$

Substituting n = 2 into equation (28) gives:

$$M_1 Z_1 \sin 3\alpha + M_1 Z_2 \sin\alpha = A_0\left[K_5 \sin 3\alpha + K_6 \sin\alpha\right]$$

Expansion of the terms K_4 and Z_2 show them to be identically zero (see Appendix A). Therefore, equating coefficients of common arguments of the sine function gives:

$$M_1 = \frac{K_5}{Z_1} A_0$$

$$A_1(x) = M_1 \cos \frac{2\pi x}{L} \qquad (29)$$

The unknown frequency is contained in the term K_6.

$$M_1 Z_2 = A_o K_6 = 0$$

$$= A_o \rho \omega_1^2 \left[1 + \Theta\left(\frac{\pi}{L}\right)^2\right] = 0$$

Therefore:

$$\omega_1^2 = 0 \tag{30}$$

Thus, the first eigen frequency has no contribution from the first order ϵ solution.

The second order terms of ϵ in equation (20) produce a linear differential equation in $A_2(x)$ with known variable coefficients.

$$A_2''' K_1 \cos\alpha + A_2'' K_2 \sin\alpha - A_2' K_3 \cos\alpha - A_2 K_4 \sin\alpha =$$
$$\xi_1 \sin\alpha + \xi_2 \sin 3\alpha + \xi_3 \sin 5\alpha \tag{31}$$

Equation (31) is similar to equation (27). To maintain constant volume, it has been required that

$$\int_0^L A_2(x)\,dx = 0$$

The solution of $A_2(x)$, therefore, will be of the form:

$$A_2(x) = \sum_{n=0}^{\infty} M_n \cos n\alpha \tag{32}$$

Substitution into the differential equation (31) gives .

The value of n = + 2, +4 are taken because cos x = cos (-x).

$$A_2(x) = M_2 \cos 2\alpha + M_3 \cos 4\alpha \tag{33}$$

This is the part of the area function whose coefficient is ϵ^2.

Substitution of equation (33) into equation (31) gives:

$$M_2 Z_2 \sin\alpha + [M_2 Z_1 + M_3 Z_3] \sin 3\alpha + M_3 Z_4 \sin 5\alpha =$$
$$\xi_1 \sin\alpha + \xi_2 \sin 3\alpha + \xi_3 \sin 5\alpha \tag{34}$$

Z_2 is identically zero (see Appendix A), therefore:

$$M_2 Z_2 = \xi_1 = 0$$

The frequency ω_2^2 is contained in the term ξ_1.

$$\xi_1 = \xi_4 - A_0 \omega_2^2 \left[\rho + \Theta \rho \left(\tfrac{\pi}{L} \right)^2 \right] = 0$$

$$\omega_2^2 = \frac{\xi_4}{A_0 \left[\rho + \Theta \rho \left(\tfrac{\pi}{L} \right)^2 \right]} \tag{35}$$

To find the unknown amplitudes, coefficients of sin 5α and sin 3α are equated.

$$M_3 = \xi_3 / Z_4$$

$$M_2 = \frac{\xi_2 - M_3 Z_3}{Z_1}$$

At this point the class of areas in terms of the parameter ϵ is determined to the second order of ϵ. ω_2^2 is the contribution to the first eigen frequency from the ϵ^2 solution.

$$A = A_0 + \epsilon M_1 \cos 2\alpha + \epsilon^2 \left[M_2 \cos 2\alpha + M_3 \cos 4\alpha \right] \tag{36}$$

The frequency of the first mode of vibration has also been determined to the second order of ϵ.

$$\omega^2 = \frac{E C_1 \left(\tfrac{\pi}{L} \right)^4 + \epsilon^2 \, \xi_4 / A_0}{\rho \left[1 + \Theta \left(\tfrac{\pi}{L} \right)^2 \right]} \tag{37}$$

Equations (27) and (31) are third order ordinary differential equations. They both have three boundary conditions imposed on them. For equation (27) the boundary conditions are:

$$x = 0 \quad ; \quad A_1 = M_1$$

$$x = L \quad ; \quad A_1 = M_1$$

$$\int_0^L A_1 \, dx = 0$$

The conditions on A_2 are:

$$x = 0 \quad ; \quad A_2 = M_2 + M_3$$

$$x = L \quad ; \quad A_2 = M_2 + M_3$$

$$\int_0^L A_2 \, dx = 0$$

The solutions obtained for A_1 and A_2 satisfy both the differential equation and the boundary conditions in both cases. These solutions are then, unique [13].

The class of areas obtained here are symmetric about the midpoint of the beam for all values of ϵ. For $\epsilon = 0$ it may be seen that the solution for the prismatic beam is obtained.

SOLUTION OF EIGEN FREQUENCIES AND MODE SHAPES

Extensive algebraic manipulations have been carried out throughout the following work. For the sake of clarity of presentation, many long algebraic expressions are represented by symbols. The expansion of these symbols may be found in Appendix A.

Using the results obtained for A_1 and A_2, it is now possible to obtain equation (20) in the form:

$$f_1(x,\epsilon)y_n^{IV} + f_2(x,\epsilon)y_n''' + f_3(x,\epsilon,\omega_n^2)y_n'' + f_4(x,\epsilon,\omega_n^2)y_n' + f_5(x,\epsilon,\omega_n^2)y_n = 0 \quad (38)$$

For the simply supported beam, the boundary conditions are:

$$x = 0 \; ; \; y = 0 \, , \; y'' = 0$$

$$x = L \; ; \; y = 0 \, , \; y'' = 0$$

The procedure in solving equation (38) will be similar to the procedure used in obtaining the variable area functions A_1 and A_2. That is, three separate differential equations in y_{n0} , y_{n1}, and y_{n2} will be obtained by equating like powers of ϵ on either side of the equation. Solutions to these three equations will give all the eigen frequencies and deflection modes in terms of x and the perturbation parameter ϵ.

The notation used will be as follows:

$$y_n = y_{n0} + \epsilon y_{n1} + \epsilon^2 y_{n2} \quad (39)$$

$$\omega_n^2 = \omega_{n0}^2 + \epsilon \omega_{n1}^2 + \epsilon^2 \omega_{n2}^2 \quad (40)$$

The first of the double subscripts refers to the eigen frequency or deflection mode. The second subscript refers to the order of ϵ.

The zero order of ϵ gives the constant cross section equation.

$$EC_1 y_{n0}^{IV} + \Theta \rho \omega_{n0}^2 y_{n0}'' - \rho \omega_{n0}^2 y_{n0} = 0 \tag{41}$$

$$(y_{n0})_{x=0} = (y_{n0})_{x=L} = (y_{n0}'')_{x=0} = (y_{n0}'')_{x=L} = 0$$

The mode shapes are of the form:

$$y_{n0} = B_{n0} \sin \lambda_n x \quad ; \quad \lambda_n = \frac{n\pi}{L} \tag{42}$$

The amplitude, B_{n0}, may be determined from initial time conditions.

Substitution of equation (42) into equation (41) gives:

$$B_{n0} \sin \lambda_n x \left[EC_1 \lambda_n^4 - \Theta \rho \omega_{n0}^2 \lambda_n^2 - \rho \omega_{n0}^2 \right] = 0$$

Therefore:

$$\omega_{n0}^2 = \frac{EC_1 \lambda_n^4}{\rho \left[1 + \Theta \lambda_n^2 \right]} \tag{43}$$

This result, for n = 1, checks with the earlier result in Section IV.

Equation (24) may now be expanded into a differential equation in y_{n1}. By equating the coefficient of ϵ in equation (38) to zero, we get:

$$Q_1 y_{n1}^{IV} + Q_2 y_{n1}'' - Q_3 y_{n1} = Q_4 \sin n\alpha + Q_5 \sin 2\alpha \cos n\alpha$$
$$- Q_6 \cos 2\alpha \sin n\alpha$$

Applying trigonometric identities:

$$Q_1 y_{n1}^{IV} + Q_2 y_{n1}'' - Q_3 y_{n1} = Q_4 \sin n\alpha \tag{44}$$
$$- \tfrac{1}{2} \left[Q_5 + Q_6 \right] \sin(n-2)\alpha + \tfrac{1}{2} \left[Q_5 - Q_6 \right] \sin(n+2)\alpha$$

where,

$$Q_4 = B_{n0} \rho \left[1 + \Theta \lambda_n^2 \right] \omega_{n1}^2$$

If Q_4 were non-zero, its contribution to the solution of equation (29) would be non-periodic [7]. Therefore, its value must be chosen such that it vanishes for all values of n.

A Fourier series is assumed for y_{n1}. The coefficients of all the cosine terms in this series must be zero because the right hand side of equation (44) contains only sine terms. After substitution of the Fourier sine series into equation (29), only three sine terms have non-zero coefficients.

$$y_{n1} = B_{n1} \sin(n-2)\alpha + C_{n1} \sin n\alpha + D_{n1} \sin(n+2)\alpha \tag{45}$$

The requirement of $Q_4 = 0$ for all n makes $C_{n1} = 0$ for all n.

Substitution of the assumed shape for y_{n1} into the differential equation (44) gives:

$$B_{n1}\left[Q_1 p^4 - Q_2 p^2 - Q_3\right]\sin(n-2)\alpha + D_{n1}\left[Q_1 q^4 - Q_2 q^2 - Q_3\right]$$
$$\sin(n+2)\alpha = Q_4 \sin\alpha + \frac{1}{2}\left[Q_5 - Q_6\right]\sin(n+2)\alpha - \frac{1}{2}\left[Q_5 + Q_6\right]\sin(n-2)\alpha$$

where,

$$p = (n-2)\frac{\pi}{L}$$

$$q = (n+2)\frac{\pi}{L}$$

When n = 1, the $\sin(n-2)\alpha$ term becomes $-\sin\alpha$.

Therefore:

$$\left[Q_4 - \frac{Q_5}{2} - \frac{Q_6}{2}\right] = 0$$

Expansion of the quantity $\frac{1}{2}\left[Q_5+Q_6\right]$ for n = 1 shows that it is identically zero (see Appendix A). Therefore:

$$Q_4 = 0$$

$$B_{n0}\rho\left[1+\Theta\left(\frac{\pi}{L}\right)^2\right]\omega_{11}^2 = 0$$

$$\omega_{11}^2 = 0$$

This confirms the earlier finding in the $A_1(x)$ solution of $\omega_1^2 = 0$.

To maintain a periodic solution, Q_4 must equal zero for all other values of n. Therefore:

$$\omega_{n1}^2 = 0 \quad , \quad n \geq 1$$

The amplitudes B_{n1} and D_{n1} are found by equating terms with the same sine function arguments. For n = 1:

$$B_{11} = \frac{\left[Q_4 - \frac{Q_5}{2} - \frac{Q_6}{2}\right]}{Q_1 p^4 - Q_2 p^2 - Q_3}$$

$$D_{11} = \frac{Q_5 - Q_6}{2\left[Q_1 q^4 - Q_2 q^2 - Q_3\right]}$$

The expansion of D_{11} shows it to be identically equal to unity. This checks the earlier substitution of:

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 \quad , \quad y_1 = 1 \sin 3\alpha \quad , \quad n = 1$$

For other n:

$$B_{n1} = \frac{-\left[Q_5 + Q_6\right]}{2\left[Q_1 p^4 - Q_2 p^2 - Q_3\right]} \quad , \quad n > 1 .$$

Table 2

n	Amplitudes		Frequency
	B_{n1}	D_{n1}	ω_{n1}^2
1	0	1	0
$n>1$	$\frac{-\left[Q_5+Q_6\right]}{2\left[Q_1 p^4 - Q_2 p^2 - Q_3\right]}$	$\frac{\left[Q_5-Q_6\right]}{2\left[Q_1 q^4 - Q_2 q^2 - Q_3\right]}$	0

ω_{n1}^2 is the contribution of the first order ϵ solution to the overall eigen frequency ω_n^2.

$$\omega_n^2 = \omega_{n0}^2 + \epsilon\,\omega_{n1}^2 + \epsilon^2 \omega_{n2}^2$$

B_{n1} and D_{n1} are the amplitudes of the sine functions which contribute to the overall equation for the mode shapes.

$$y_n = y_{n0} + \epsilon y_{n1} + \epsilon^2 y_{n2}$$

where,

$$y_{n1} = B_{n1} \sin(n-2)\alpha + D_{n1} \sin(n+2)\alpha$$

The second order ϵ equation in the unknown y_{n2} is:

$$A_0^2\left[Q_1 y_{n2}^{IV} + Q_2 y_{n2}'' - Q_3 y_{n2}\right] = \delta_1 \sin(n+2)\alpha + \delta_2 \sin(n-2)\alpha$$

$$+ \delta_3 \sin(n+4)\alpha + \delta_4 \sin(n-4)\alpha + \delta_5 \sin n\alpha \qquad (46)$$

The assumed shape of y_{n2} is therefore:

$$y_{n2} = B_{n2} \sin(n+2)\alpha + D_{n2} \sin(n-2)\alpha + E_{n2} \sin(n+4)\alpha$$

$$+ F_{n2} \sin(n-4)\alpha + G_{n2} \sin n\alpha$$

The term δ_5 is:

$$\delta_5 = B_{n0}\, \rho\left[1 + \Theta \lambda_n^2\right] \omega_{n2}^2 + \delta_6$$

As was the case with the first order ϵ solution, this term may not be allowed to persist if the solution is to be periodic. This condition defines the frequency and demands that $G_{n2} = 0$.

For $n = 1$, the $\delta_2 \sin(n-2)\alpha$ term becomes $- \delta_2 \sin\alpha$.

Therefore:

$$\omega_{12}^2 = \frac{\delta_2 - \delta_6}{B_{10}\, \rho\left[1 + \Theta\left(\frac{\pi}{L}\right)^2\right] A_0^2}$$

Similarly for $n = 2$, the $\delta_4 \sin(n-4)\alpha$ term becomes $- \delta_4 \sin 2\alpha$ and combines with the $\delta_5 \sin 2\alpha$ term. Therefore:

$$\omega_{22}^2 = \frac{\delta_4 - \delta_6}{B_{20}\, \rho\left[1 + 4\Theta\left(\frac{\pi}{L}\right)^2\right] A_0^2}$$

The amplitudes B_{n2}, D_{n2}, E_{n2}, and F_{n2} are found, as in the first order ϵ solution, by equating coefficients of the sine functions with like arguments. Care must be taken, however, noting combinations for certain values of n. Substitution of the assumed form of y_{n2} into the differential equation (46) gives:

$$B_{n2}\left[Q_1 q^4 - Q_2 q^2 - Q_3\right]\sin(n+2)\alpha + D_{n2}\left[Q_1 p^4 - Q_2 p^2 - Q_3\right]$$
$$\sin(n-2)\alpha + E_{n2}\left[Q_1 q'^4 - Q_2 q'^2 - Q_3\right]\sin(n+4)\alpha +$$
$$F_{n2}\left[Q_1 p'^4 - Q_2 p'^2 - Q_3\right]\sin(n-4)\alpha = \left[\delta_1 \sin(n+2)\alpha + \delta_2 \sin(n-2)\alpha\right.$$
$$\left. + \delta_3 \sin(n+4)\alpha + \delta_4 \sin(n-4)\alpha + \delta_5 \sin n\alpha\right]\frac{1}{A_o^2}$$

where,

$$p' = (n-4)\frac{\pi}{L}$$

$$q' = (n+4)\frac{\pi}{L}$$

For n = 1

$$\left[Q_1\left(\frac{3\pi}{L}\right)^4 - Q_2\left(\frac{3\pi}{L}\right)^2 - Q_3\right]\left[B_{12} - F_{12}\right] = (\delta_1 - \delta_4)\frac{1}{A_o^2}$$

The expansion of $\delta_1 - \delta_4$ shows it to be identically zero for n = 1 (see Appendix A). The expression $\left[Q_1\left(\frac{3\pi}{L}\right)^4 - Q_2\left(\frac{3\pi}{L}\right)^2 - Q_3\right]$ is non-zero. Therefore $[B_{12} - F_{12}]$, which is the coefficient of sin 3α for n = 1, is zero.

$$D_{12}\left[Q_1\left(\frac{\pi}{L}\right)^4 - Q_2\left(-\frac{\pi}{L}\right)^2 - Q_3\right] = (\delta_5 - \delta_2)\frac{1}{A_o^2}$$

From the earlier calculation of ω_{12}^2, we know that $[\delta_5 - \delta_2] = 0$.

Therefore:

$$D_{12} = 0$$

$$E_{12}\left[Q_1\left(\frac{5\pi}{L}\right)^4 - Q_2\left(\frac{5\pi}{L}\right)^2 - Q_3\right] = \delta_3 / A_o^2$$

Expansion of $\delta_3 / \left[Q_1 q'^4 - Q_2 q'^2 - Q_3\right] A_o^2$ shows it to be identically equal to one for n = 1 (see Appendix A).

$$E_{12} = 1$$

This checks the earlier substitution, the the A_2 solution, of:

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 \quad , y_2 = 1 \sin 5\frac{\pi x}{L} \quad , n = 1$$

Expansion of the expression for ω_{12}^2 also shows it to be consistent for the value ω_2^2 found in the A_2 solution.

For n = 2 the sin(n-2)α term vanishes. Similarly, for n = 4 the sin(n-4)α term vanishes. Therefore:

$$D_{22} = 0$$

$$F_{42} = 0$$

For all other n.

$$B_{n2} = \frac{\delta_1}{[Q_1 q^4 - Q_2 q^2 - Q_3] A_0^2} \quad , n > 1$$

$$D_{n2} = \frac{\delta_2}{[Q_1 p^4 - Q_2 p^2 - Q_3] A_0^2} \quad , n > 2$$

$$E_{n2} = \frac{\delta_3}{[Q_1 q'^4 - Q_2 q'^2 - Q_3] A_0^2} \quad , n \geq 1$$

$$F_{n2} = \frac{\delta_4}{[Q_1 p'^4 - Q_3 p'^2 - Q_3] A_0^2} \quad , n \neq 4, n \neq 1$$

ω_{n2}^2 is the contribution of the second order ϵ solution to the overall eigen frequency ω_n^2.

$$\omega_n^2 = \omega_{n0}^2 + \epsilon \omega_{n1}^2 + \epsilon^2 \omega_{n2}^2$$

B_{n2}, D_{n2}, E_{n2}, and F_{n2}, are the amplitudes of the sine functions which contribute to the overall equation for the mode shapes.

$$y_n = y_{n0} + \epsilon y_{n1} + \epsilon^2 y_{n2}$$

where,

$$y_{n2} = B_{n2} \sin(n+2)\alpha + D_{n2} \sin(n-2)\alpha + E_{n2} \sin(n+4)\alpha + F_{n2} \sin(n-4)\alpha$$

The mode shapes for the class of areas derived in Section IV are:

$$y_n = B_{n0} \sin \frac{n\pi x}{L} + \epsilon \left[B_{n1} \sin px + D_{n1} \sin qx \right] +$$

$$\epsilon^2 \left[B_{n2} \sin qx + D_{n2} \sin px + E_{n2} \sin q'x + F_{n2} \sin p'x \right] \qquad (47)$$

Table 3

n	Amplitudes				Frequency
	B_{n2}	D_{n2}	E_{n2}	F_{n2}	ω_{n2}^2
1	0	0	1	0	$\dfrac{(\delta_2-\delta_6)_{n=1}}{B_{10}\rho[1+\Theta(\frac{\pi}{L})^2]A_0^2}$
2	$\dfrac{\delta_1/A_0^2}{[Q_1(\frac{4\pi}{L})^4-Q_2(\frac{4\pi}{L})^2-Q_3]}$	0	$\dfrac{\delta_3/A_0^2}{[Q_1(\frac{6\pi}{L})^4-Q_2(\frac{6\pi}{L})^2-Q_3]}$	$\dfrac{\delta_4/A_0^2}{[Q_1(\frac{2\pi}{L})^4-Q_2(\frac{2\pi}{L})^2-Q_3]}$	$\dfrac{\delta_4-\delta_6}{B_{20}\rho[1+4\Theta(\frac{\pi}{L})^2]A_0^2}$
3	$\dfrac{\delta_1/A_0^2}{[Q_1(\frac{5\pi}{L})^4-Q_2(\frac{5\pi}{L})^2-Q_3]}$	$\dfrac{\delta_2/A_0^2}{[Q_1(\frac{\pi}{L})^4-Q_2(\frac{\pi}{L})^2-Q_3]}$	$\dfrac{\delta_3/A_0^2}{[Q_1(\frac{7\pi}{L})^4-Q_2(\frac{7\pi}{L})^2-Q_3]}$	$\dfrac{\delta_4/A_0^2}{[Q_1(\frac{\pi}{L})^4-Q_2(\frac{\pi}{L})^2-Q_3]}$	$\dfrac{-\delta_6}{B_{30}\rho[1+9\Theta(\frac{\pi}{L})^2]A_0^2}$
4	$\dfrac{\delta_1/A_0^2}{[Q_1(\frac{6\pi}{L})^4-Q_2(\frac{6\pi}{L})^2-Q_3]}$	$\dfrac{\delta_2/A_0^2}{[Q_1(\frac{2\pi}{L})^4-Q_2(\frac{2\pi}{L})^2-Q_3]}$	$\dfrac{\delta_3/A_0^2}{[Q_1(\frac{8\pi}{L})^4-Q_2(\frac{8\pi}{L})^2-Q_3]}$	0	$\dfrac{-\delta_6}{B_{40}\rho[1+16\Theta(\frac{\pi}{L})^2]A_0^2}$
$n>4$	$\dfrac{\delta_1/A_0^2}{[Q_1q^4-Q_2q^2-Q_3]}$	$\dfrac{\delta_2/A_0^2}{[Q_1p^4-Q_2p^2-Q_3]}$	$\dfrac{\delta_3/A_0^2}{[Q_1q'^4-Q_2q'^2-Q_3]}$	$\dfrac{\delta_4/A_0^2}{[Q_1p'^4-Q_2p'^2-Q_3]}$	$\dfrac{-\delta_6}{B_{n0}\rho[1+\Theta\lambda_n^2]A_0^2}$

The eigen frequencies are:

$$\omega_n = \sqrt{\frac{B_{n0} E C_1 \lambda_n^4 + \epsilon^2 \chi_n}{B_{n0} \rho [1 + \Theta \lambda_n^2]}} \tag{48}$$

where,

$$\chi_1 = (\delta_2 - \delta_6)_{n=1} \frac{1}{A_0^2}$$

$$\chi_2 = (\delta_4 - \delta_6)_{n=2} \frac{1}{A_0^2}$$

$$\chi_n = -\delta_6 \frac{1}{A_0^2} \quad , \; n > 2$$

B_{n0} is an amplitude which depends on the initial time conditions of the problem.

By setting $\epsilon = 0$ in equations (47) and (48), the solution is reduced to the constant cross section solution. This is expected because when $\epsilon = 0$:

$$A = A_0 = \text{constant}$$

By setting $n = 1$ in equations (47) and (48), the values of and ω_1 are:

$$y_1 = \sin \frac{\pi x}{L} + \epsilon \sin 3\frac{\pi x}{L} + \epsilon^2 \sin 5\frac{\pi x}{L}$$

$$\omega_1^2 = \frac{E C_1 (\frac{\pi}{L})^4 + \epsilon^2 \chi_1}{\rho [1 + \Theta (\frac{\pi}{L})^2]}$$

These results are identical with the results and substitutions from Section IV, the solution for the area function. This checks the great amount of algebra which was performed in obtaining formulas for the eigen frequencies and mode shapes.

CONCLUSIONS

A fundamental non-linear ordinary differential equation, equation (16), has been derived for the flexural vibrations of a non-uniform beam, including shear.

Equation (20) is the particular form taken by equation (16) under the specializing assumptions that moment of inertia is directly proportional to cross sectional area and that the shear coefficient is constant.

Tables 2 and 3 contain approximate solutions of equation (20) for all eigen frequencies and mode shapes in terms of a perturbation parameter ϵ. The perturbation solutions are accurate up to the second order of ϵ.

EXTENSIONS

The preceding results could be applied to beam optimization design. Using the formulas of the preceding sections, the deflection may be obtained as:

$$y = f(x, t, \epsilon)$$

It may be desirable to minimize the deflection y. For example, minimization of the amplitude of vibration could cut down on noise level. Bending stress is a function of the deflection and this quantity could also be minimized. This minimization process could be approached through iteration on the perturbation parameter ϵ. A value of ϵ would be chosen and $y(x,t,\epsilon)$ would be evaluated at increments over the beam length over appropriate time intervals. The maximum value obtained from this process would be saved. This process would then be repeated for other values of ϵ. The minimum value of the maximums noted for each iteration would give the optimum value for ϵ.

It must be noted that the equations developed are valid only for small values of ϵ.

Another possibility for further study would be the application of this method to the forced vibration case and subsequent optimization of design for deflection, stress, or frequency.

An experimental investigation to compare with theoretical results would be useful. Problems would probably arise in the manufacture of of the beam itself and the development of devices sensitive and accurate enough to detect variations from the solution for a prismatic beam.

APPENDIX A

EXPANSION OF SYMBOLS

This Appendix contains the expansion of the symbols used in the main body of the thesis and those symbols used in Appendix D. Symbols equal to zero are shown.

The expansion of the symbols used in the area solutions follows.

$$K_1 = \Theta^2 \rho \omega_0^2 \left(\frac{\pi}{L}\right) - \Theta E C_1 \left(\frac{\pi}{L}\right)^3$$

$$K_2 = 3\Theta \rho \omega_0^2 + \Theta^2 \rho \omega_0^2 \left(\frac{\pi}{L}\right)^2 + E C_1 \left(\frac{\pi}{L}\right)^2 - \Theta E C_1 \left(\frac{\pi}{L}\right)^4$$

$$K_3 = -2 E C_1 \left(\frac{\pi}{L}\right)^3$$

$$K_4 = 2\left[E C_1 \left(\frac{\pi}{L}\right)^4 - \rho \omega_0^2 - \Theta \rho \omega_0^2 \left(\frac{\pi}{L}\right)^2\right] = 0$$

$$K_5 = E C_1 \left(\frac{3\pi}{L}\right)^4 - \rho \omega_0^2 - \Theta \rho \omega_0^2 \left(\frac{3\pi}{L}\right)^2$$

$$K_6 = -\Theta \rho \omega_1^2 \left(\frac{\pi}{L}\right)^2 - \rho \omega_1^2 = 0 \quad , (\omega_1^2 = 0)$$

$$K_7 = -\rho \omega_2^2 - \Theta \rho \omega_2^2 \left(\frac{\pi}{L}\right)^2$$

$$K_8 = -\rho \omega_1^2 - \Theta \rho \omega_1^2 \left(\frac{3\pi}{L}\right)^2 = 0$$

$$K_9 = E C_1 \left(\frac{5\pi}{L}\right)^4 - \rho \omega_0^2 - \Theta \rho \omega_0^2 \left(\frac{5\pi}{L}\right)^2$$

$$K_{10} = 2 K_6 = 0$$

$$K_{11} = -2\Theta^2 \rho \omega_0^2 - \Theta E C_1 \left(\frac{\pi}{L}\right)^2$$

$$K_{12} = 9\Theta^2 \rho \omega_0^2 \left(\frac{\pi}{L}\right) - 6\Theta E C_1 \left(\frac{\pi}{L}\right)^3$$

$$K_{13} = \Theta^2 \rho \omega_0^2 \left(\frac{\pi}{L}\right)^2 - \Theta E C_1 \left(\frac{\pi}{L}\right)^4 + 7\Theta \rho \omega_0^2$$

$$K_{14} = -\Theta E C_1 \left(\frac{\pi}{L}\right)^2 - \Theta^2 \rho \omega_0^2$$

$$K_{15} = -3\Theta \rho \omega_1^2 - \Theta^2 \rho \omega_1^2 \left(\frac{\pi}{L}\right)^2 = 0$$

$$K_{16} = \Theta E C_1 \left(\frac{3\pi}{L}\right)^4 - 3\Theta \rho \omega_0^2 - \Theta^2 \rho \omega_0^2 \left(\frac{3\pi}{L}\right)^2 - E C_1 \left(\frac{3\pi}{L}\right)^2$$

$$K_{17} = \Theta^2 \rho \omega_0^2 \left(\frac{3\pi}{L}\right) - \Theta E C_1 \left(\frac{3\pi}{L}\right)^3$$

$$K_{18} = \Theta^2 \rho \omega_1^2 \left(\frac{\pi}{L}\right) = 0$$

$$Z_1 = 4K_1\left(\frac{\pi}{L}\right)^3 - 2K_2\left(\frac{\pi}{L}\right)^2 + K_3\left(\frac{\pi}{L}\right)$$
$$Z_2 = 4K_1\left(\frac{\pi}{L}\right)^3 + 2K_2\left(\frac{\pi}{L}\right)^2 + K_3\left(\frac{\pi}{L}\right) = 0$$
$$Z_3 = 32K_1\left(\frac{\pi}{L}\right)^3 + 8K_2\left(\frac{\pi}{L}\right)^2 + 2K_3\left(\frac{\pi}{L}\right)$$
$$Z_4 = 32K_1\left(\frac{\pi}{L}\right)^3 - 8K_2\left(\frac{\pi}{L}\right)^2 + 2K_3\left(\frac{\pi}{L}\right)$$
$$\beta_1 = A_0 K_7$$
$$\beta_2 = A_0 K_8 = 0$$
$$\beta_3 = A_0 K_9$$
$$\beta_4 = N_1 A_0\left[K_{10} - \left(\frac{2\pi}{L}\right)^2 K_{15}\right], \quad N_1 = \frac{K_5}{Z_1}$$
$$\beta_5 = N_1 A_0\left[2K_5 - \left(\frac{2\pi}{L}\right)^2 K_{16}\right]$$
$$\beta_6 = N_1 A_0\left[54\left(\frac{\pi}{L}\right)K_3 + \left(\frac{2\pi}{L}\right)^3 K_{17}\right]$$
$$\beta_7 = N_1^2 A_0\left[\left(\frac{2\pi}{L}\right)^3 K_{12} - \left(\frac{2\pi}{L}\right)^3 K_1 - \left(\frac{2\pi}{L}\right)K_3\right]$$
$$\beta_8 = N_1^2 A_0\left[\left(\frac{2\pi}{L}\right)^4 K_{11} + \left(\frac{2\pi}{L}\right)^2 K_2\right]$$
$$\beta_9 = N_1^2 A_0\left[\left(\frac{2\pi}{L}\right)^2 K_{13} + \left(\frac{2\pi}{L}\right)^4 K_{14}\right]$$
$$\beta_{10} = N_1 A_0\left[\left(\frac{2\pi}{L}\right)^3 K_{18}\right] = 0$$
$$\xi_1 = \frac{1}{2}\left[2\beta_1 - \beta_4 + \beta_5 + \beta_6 + \beta_8 + \beta_9\right]$$
$$\xi_2 = \frac{1}{2}\left[2\beta_2 + \beta_4 + \beta_7/2 - \beta_8/2 + \beta_9/2\right]$$
$$\xi_3 = \frac{1}{2}\left[2\beta_3 + \beta_5 - \beta_6 + \beta_7/2 + \beta_8/2 - \beta_9/2\right]$$
$$\xi_4 = -\frac{1}{2}\left[\beta_4 - \beta_5 - \beta_6 - \beta_8 - \beta_9\right]$$
$$M_1 = \frac{K_5}{Z_1}A_0$$
$$M_2 = \frac{\xi_2 - M_3 Z_3}{Z_1}$$
$$M_3 = \xi_3/Z_4$$

The following symbols were used in the solution for the eigen frequencies and mode shapes.

$$Q_1 = EC_1$$
$$Q_2 = \Theta \rho \omega_{no}^2$$
$$Q_3 = \rho \omega_{no}^2$$
$$Q_4 = B_{no}[\rho + \Theta \rho \lambda_n^2]\omega_{n1}^2 = 0 \quad , \omega_{n1}^2 = 0 \quad , \lambda_n = n\alpha$$
$$Q_5 = B_{no} N_1 (\tfrac{2\pi}{L})[\Theta^2 \rho \omega_{no}^2 \lambda_n (\tfrac{2\pi}{L})^2 - \Theta EC_1 \lambda_n^3 (\tfrac{2\pi}{L})^2 - 2EC_1 \lambda_n^3]$$
$$Q_6 = B_{no} N_1 (\tfrac{2\pi}{L})^2 [3\Theta \rho \omega_{no}^2 - \Theta EC_1 \lambda_n^4 + \Theta^2 \rho \omega_{no}^2 \lambda_n^2 + EC_1 \lambda_n^2]$$
$$Q_7 = B_{no}[EC_1 \lambda_n^4 - \rho \omega_{no}^2 - \Theta \rho \omega_{no}^2 \lambda_n^2] = 0$$
$$Q_8 = B_{no}[2EC_1 \lambda_n^3]$$
$$Q_9 = B_{no}[3\Theta \rho \omega_{no}^2 - \Theta EC_1 \lambda_n^4 + \Theta^2 \rho \omega_{no}^2 \lambda_n^2 + EC_1 \lambda_n^2]$$
$$Q_{10} = B_{no}[\Theta EC_1 \lambda_n^3 - \Theta^2 \rho \omega_{no}^2 \lambda_n]$$
$$Q_{11} = B_{no}[\rho + \Theta \rho \lambda_n^2]\omega_{n2}^2$$
$$Q_{12} = B_{n1}[\rho + \Theta \rho p^2]\omega_{n1}^2 = 0 \quad , p = (n-2)\tfrac{\pi}{L}$$
$$Q_{13} = D_{n1}[\rho + \Theta \rho q^2]\omega_{n1}^2 = 0 \quad , q = (n+2)\tfrac{\pi}{L}$$
$$Q_{14} = B_{n1}[EC_1 p^4 - \rho \omega_{no}^2 - \Theta \rho \omega_{no}^2 p^2]$$
$$Q_{15} = D_{n1}[EC_1 q^4 - \rho \omega_{no}^2 - \Theta \rho \omega_{no}^2 q^2]$$
$$Q_{16} = B_{no}[\rho + \Theta \rho \lambda_n^2]\omega_{n1}^2 = 0$$
$$Q_{17} = B_{n1}[2EC_1 p^3]$$
$$Q_{18} = D_{n1}[2EC_1 q^3]$$
$$Q_{19} = Q_8$$
$$Q_{20} = B_{no}[2\Theta^2 \rho \omega_{no}^2 + \Theta EC_1 \lambda_n^2]$$
$$Q_{21} = B_{no}[9\Theta^2 \rho \omega_{no}^2 \lambda_n - 6\Theta EC_1 \lambda_n^3]$$
$$Q_{22} = B_{no}[\Theta EC_1 \lambda_n^4 - \Theta^2 \rho \omega_{no}^2 \lambda_n^2 + 7\Theta \rho \omega_{no}^2]$$

$$Q_{23} = B_{n0}[\Theta E C_1 \lambda_n^2 + \Theta^2 \rho \omega_{n0}^2]$$

$$Q_{24} = B_{n1}[\Theta E C_1 p^4 - 3\Theta\rho\omega_{n0}^2 - \Theta^2\rho\,\omega_{n0}^2 p^2 - EC_1 p^2]$$

$$Q_{25} = D_{n1}[\Theta E C_1 q^4 - 3\Theta\rho\omega_{n0}^2 - \Theta^2\rho\omega_{n0}^2 q^2 - EC_1 q^2]$$

$$Q_{26} = B_{n0}[3\Theta\rho\omega_{n1}^2 + \Theta^2\rho\omega_{n1}^2\,\lambda_n^2] = 0$$

$$Q_{27} = -Q_9$$

$$Q_{28} = B_{n1}[-\Theta E C_1 p^3 + \Theta^2\rho\,\omega_{n0}^2 p]$$

$$Q_{29} = D_{n1}[-\Theta E C_1 q^3 + \Theta^2\rho\omega_{n0}^2 q]$$

$$Q_{30} = B_{n0}[\Theta^2\rho\omega_{n1}^2\lambda_n] = 0$$

$$Q_{31} = -Q_{10}$$

$$\phi_1 = -A_0 M_2\left[\left(\tfrac{2\pi}{L}\right)^2 Q_9\right]$$

$$\phi_2 = -A_0 M_3\left[\left(\tfrac{4\pi}{L}\right)^2 Q_9\right]$$

$$\phi_3 = -A_0 M_2\left[\left(\tfrac{2\pi}{L}\right) Q_8 + \left(\tfrac{2\pi}{L}\right)^3 Q_{10}\right]$$

$$\phi_4 = A_0 M_3\left[\left(\tfrac{4\pi}{L}\right) Q_8 + \left(\tfrac{4\pi}{L}\right)^3 Q_{10}\right]$$

$$\phi_5 = A_0^2 Q_{11}$$

$$\phi_6 = A_0^2 Q_{12}$$

$$\phi_7 = A_0^2 Q_{13}$$

$$\phi_8 = A_0^2 N_1\left[2Q_{14} - \left(\tfrac{2\pi}{L}\right)^2 Q_{24}\right]$$

$$\phi_9 = A_0^2 N_1\left[2Q_{15} - \left(\tfrac{2\pi}{L}\right)^2 Q_{25}\right]$$

$$\phi_{10} = A_0^2 N_1\left[\left(\tfrac{2\pi}{L}\right)^3 Q_{28} - \left(\tfrac{2\pi}{L}\right) Q_{17}\right]$$

$$\phi_{11} = A_0^2 N_1\left[\left(\tfrac{2\pi}{L}\right)^3 Q_{29} - \left(\tfrac{2\pi}{L}\right) Q_{18}\right]$$

$$\phi_{12} = A_0^2 N_1^2\left[\left(\tfrac{2\pi}{L}\right)^3 Q_{31} - \left(\tfrac{2\pi}{L}\right) Q_{19} - \left(\tfrac{2\pi}{L}\right)^3 Q_{21}\right]$$

$$\phi_{13} = A_0^2 N_1^2\left[\left(\tfrac{2\pi}{L}\right)^4 Q_{20} + \left(\tfrac{2\pi}{L}\right)^2 Q_{27}\right]$$

$$\phi_{14} = A_0^2 N_1^2\left[\left(\tfrac{2\pi}{L}\right)^4 Q_{23} - \left(\tfrac{2\pi}{L}\right)^2 Q_{22}\right]$$

$$\delta_1 = \frac{1}{2}\left[\phi_1 + \phi_3 + \phi_7\right]$$

$$\delta_2 = \frac{1}{2}\left[\phi_1 - \phi_3 + \phi_6\right]$$

$$\delta_3 = \frac{1}{2}\left[\phi_2 - \phi_4 - \phi_9 + \phi_{11} + \phi_{12}/2 + \phi_{13}/2 - \phi_{14}/2\right]$$

$$\delta_4 = \frac{1}{2}\left[\phi_2 + \phi_4 - \phi_8 - \phi_{10} - \phi_{12}/2 + \phi_{13}/2 - \phi_{14}/2\right]$$

$$\delta_5 = \frac{1}{2}\left[2\phi_5 - \phi_8 - \phi_9 + \phi_{10} - \phi_{11} + \phi_{13} + \phi_{14}\right]$$

$$\delta_6 = \frac{1}{2}\left[-\phi_8 - \phi_9 + \phi_{10} - \phi_{11} + \phi_{13} + \phi_{14}\right]$$

$$p' = (n-4)\frac{\pi}{L}$$

$$q' = (n+4)\frac{\pi}{L}$$

APPENDIX B

THE CASE WHEN B* IS ZERO

On page 10 an assumption that B* was not equal to zero was made. Suppose B* did vanish:

$$B^* = 1 - [EI\sigma']' = 0$$

$$[EI\sigma']' = 1$$

Integrating with respect to x:

$$EI \frac{d\sigma}{dx} = x + a$$

If k = constant and I = $C_1 A$

$$\sigma = \frac{1}{kAG} \quad , \sigma' = -\frac{1}{kG} \frac{A'}{A^2}$$

$$-\frac{EC_1}{kG} \int_0^A \frac{dA}{A} = \int_0^x (x+a)\,dx$$

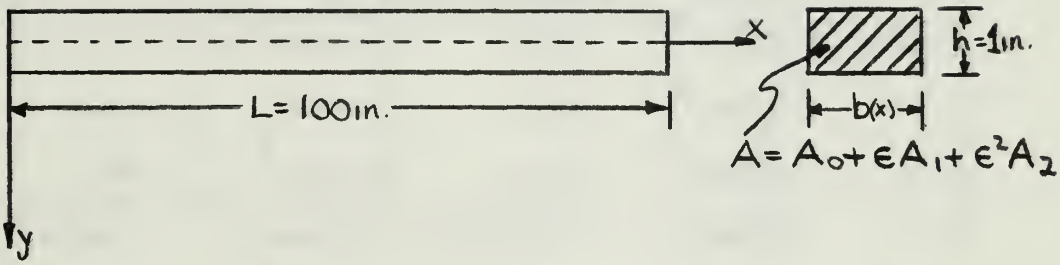
$$-\frac{EC_1}{kG} \ln A = \frac{x^2}{2} + ax + b$$

$$A = \exp\left[-\frac{kG}{EC_1}\left(\frac{x^2}{2} + ax + b\right)\right]$$

This is not a symmetric function about the beam midpoint. Since we are restricting consideration to situations having symmetry about x = L/2, this case is excluded.

APPENDIX C

NUMERICAL EXAMPLE



Steel Beam

$E = 30\times10^6$ psi

$G = 12\times10^6$ psi

$k = .67$

$\rho = 490$ lbf/ft^3

$A_o = 1$ in.^{2}

$B_{no} = 1$ in., $n \geq 1$

$I = C_1 A$, $C_1 = h^2/12 = 1/12$ in.^{2}

A computer program was prepared that computes the first ten eigen frequencies and mode shapes for $\epsilon = .05$. The program also computes and plots the variable area function $\epsilon A_1 + \epsilon^2 A_2$ for $\epsilon = .01, .02, .03, .04$ and $.05$. The first ten eigen functions for $\epsilon = .05$ are also plotted.

The physical significance of $\epsilon A_1 + \epsilon^2 A_2$ in this case would be:

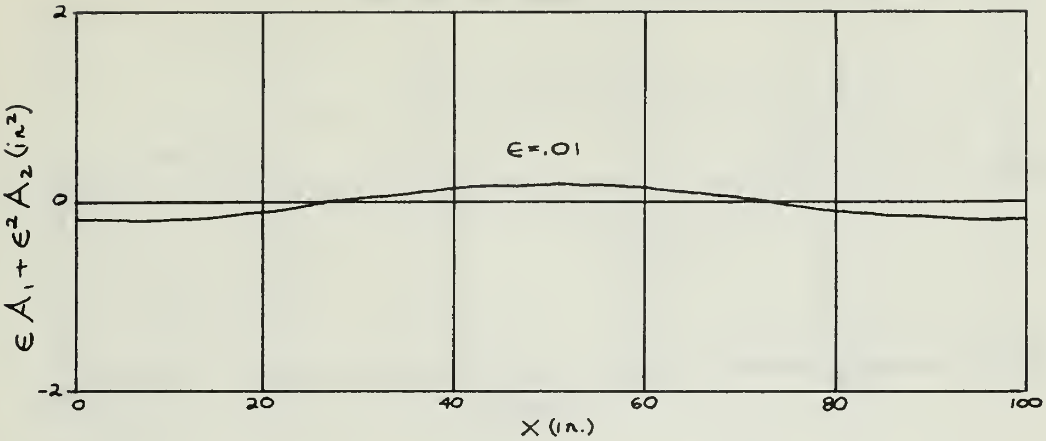
$$A = b(x) h = A_o + \epsilon A_1 + \epsilon^2 A_2$$

$$b(x) = \frac{A_o}{h} + \epsilon \frac{A_1}{h} + \epsilon^2 \frac{A_2}{h}$$

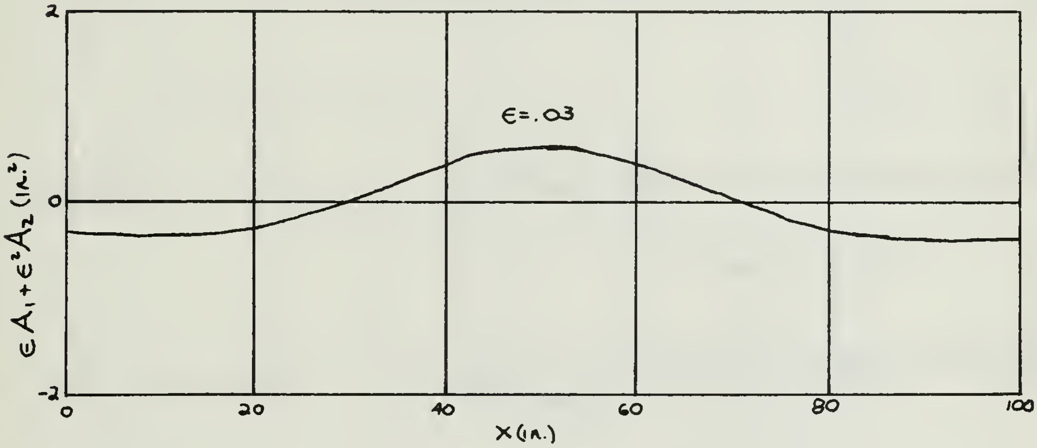
Therefore, the width would be a function of x while the height, h, remains constant.

The first ten eigen frequencies are:

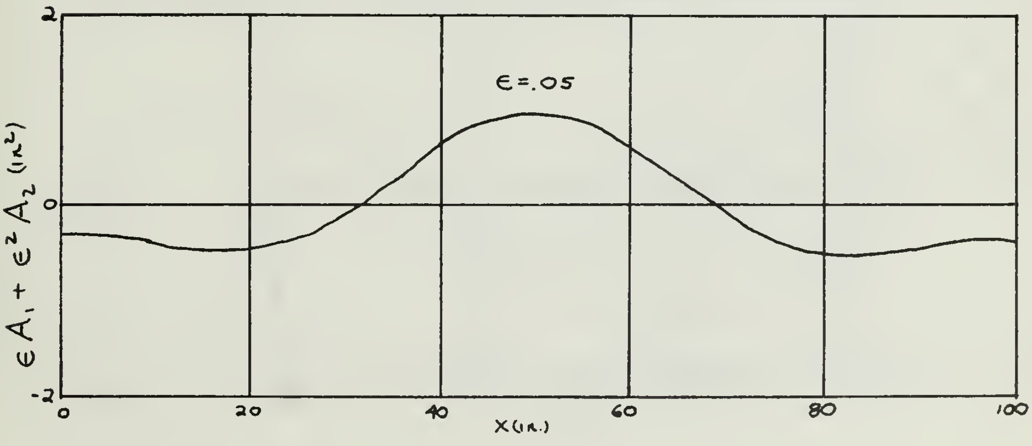
n	ω_{n0}^2	ω_{n1}^2	ω_{n2}^2	$\omega_n = \sqrt{\omega_{n0}^2 + \epsilon^2 \omega_{n2}^2}$
1	$.332 \times 10^4$	0.0	$-.261 \times 10^6$	51.6 cycles/sec.
2	$.530 \times 10^5$	0.0	$-.310 \times 10^7$	212.7 "
3	$.268 \times 10^6$	0.0	$.909 \times 10^7$	539.2 "
4	$.845 \times 10^6$	0.0	$.190 \times 10^8$	944.9 "
5	$.206 \times 10^7$	0.0	$.322 \times 10^8$	1,462.4 "
6	$.425 \times 10^7$	0.0	$.488 \times 10^8$	2,091.7 "
7	$.785 \times 10^7$	0.0	$.693 \times 10^8$	2,832.4 "
8	$.133 \times 10^8$	0.0	$.941 \times 10^8$	3,683.1 "
9	$.212 \times 10^8$	0.0	$.124 \times 10^9$	4,642.4 "
10	$.322 \times 10^8$	0.0	$.158 \times 10^9$	5,708.9 "



$\epsilon A_1 + \epsilon^2 A_2$ (in²)
ϵ=.01
2
0
-2
0
20
40
60
80
100
X (in.)

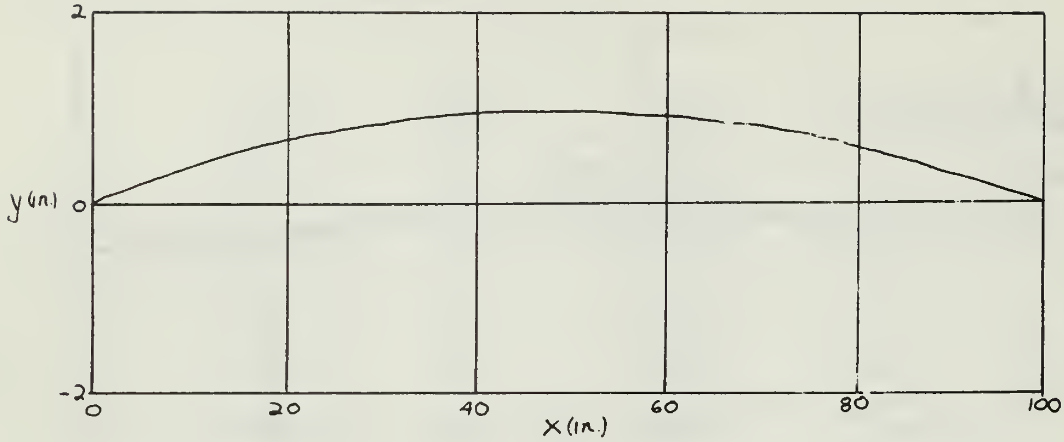


$\epsilon A_1 + \epsilon^2 A_2$ (in²)
ϵ=.03
2
0
-2
0
20
40
60
80
100
X (in.)

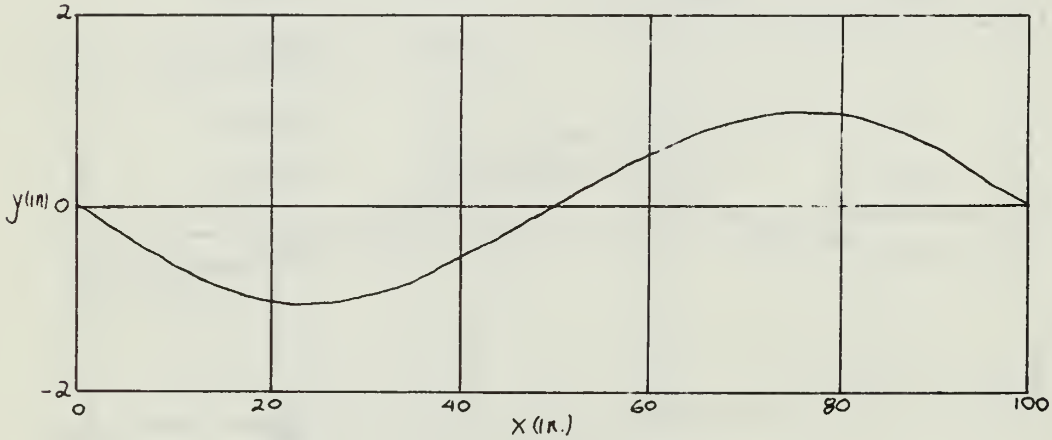


$\epsilon A_1 + \epsilon^2 A_2$ (in²)
ϵ=.05
2
0
-2
0
20
40
60
80
100
X (in.)

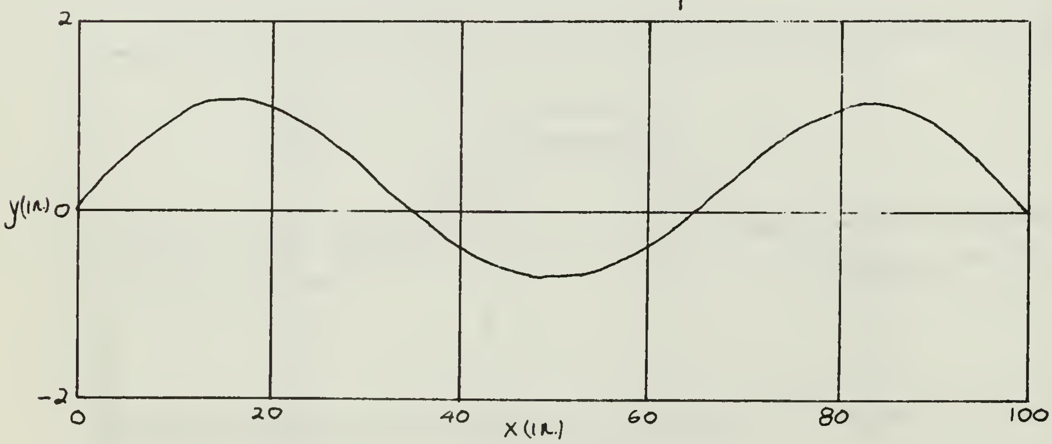
First mode shape



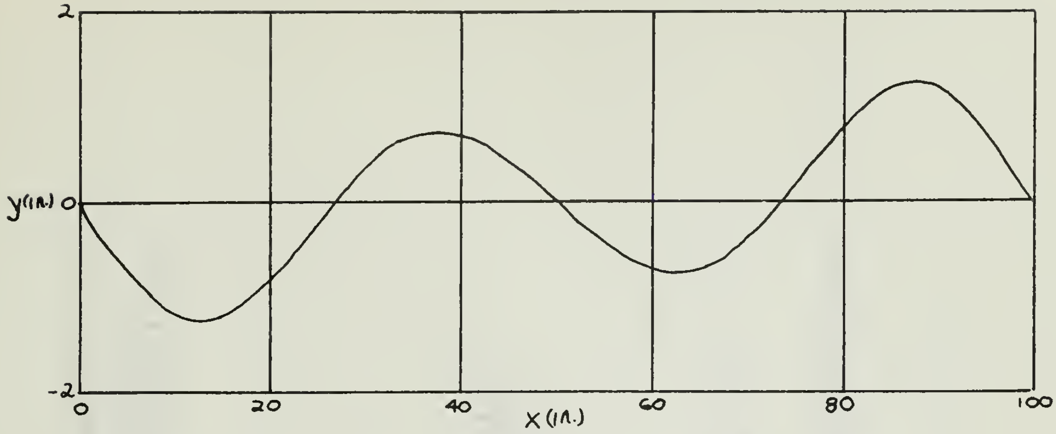
Second mode shape



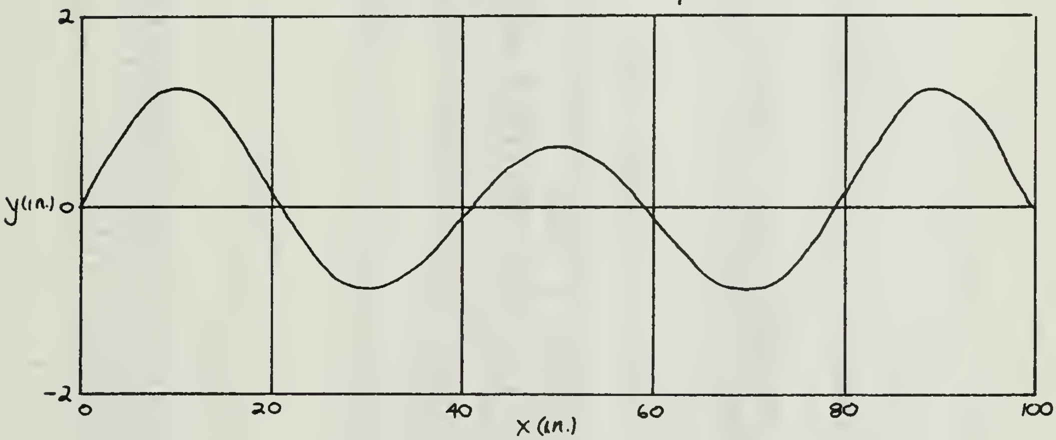
Third mode shape



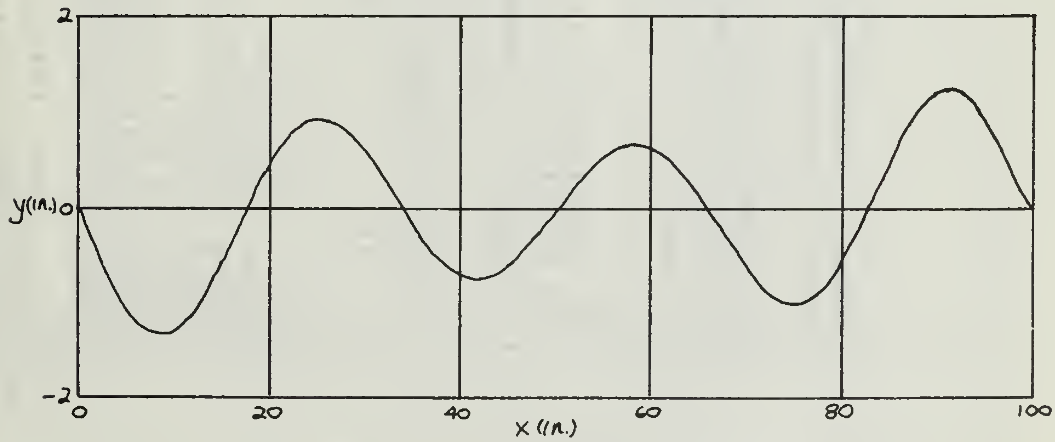
Fourth mode shape



Fifth mode shape



Sixth mode shape



```
C
C     ******************************************************************
C     EXAMPLE NUMERICAL SOLUTION
C
C     BEAM IS STEEL WITH YOUNG'S MODULUS EQUAL TO 30,000,000
C     THE SHEAR MODULUS IS 12,000,000 AND THE SHEAR COEFFICIENT IS
C     TIMOSHENKO'S VALUE FOR A RECTANGULAR CROSS SECTION .67
C
C     THE BEAM LENGTH IS 100 INCHES
C
C     THE BASIC CROSS SECTIONAL AREA IS 1 SQUARE INCH
C
C     THE AMPLITUDE OF THE MODE SHAPES FOUND THROUGH INITIAL
C     TIME CONDITIONS IS ASSUMED TO BE 1 INCH
C
C     THIS PROGRAM COMPUTES  AND PLOTS THE VARIABLE AREA FUNCTION
C     FOR FIVE DIFFERENT VALUES OF EPSILON .01,.02,.03,.04, AND .05
C
C     IT ALSO COMPUTES AND PLOTS THE FIRST TEN MODE SHAPES
C     FOR EPSILON EQUAL TO .05
C
C     THE FIRST TEN EIGEN FREQUENCIES ARE ALSO CALCULATED
C
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 LABEL/8H        /
      REAL*8 ITITLE(12)
      REAL*4 LABL1/4H    /
      REAL*4 WS(102),DEFTS(102)
      REAL*4 XYZ(102)
      REAL*4 TA(102)
      DIMENSION C(18),B(10),ZETA(4),Z(6),BN(10),DN(10),XX(10),YY(10),OM(
     13),EN(3),Q(31),ZZ(10),A(10),O(10),OC(10),OOO(10),P(15),DEL(6),
     2BT(10),CT(10),ET(10),FT(10),BB(10),U(11)
      DIMENSION DEF(2),DEFT(102),W(102),AT(102),EPS(6),H(2),OMT(10),OX(1
     10)
      READ(5,600) (ITITLE(I),I=1,12)
  600 FORMAT(6A8)
      PI=3.141592653589793
      CAY=.67
      RO=490./(32.2*12.**4.)
```

```
E=30000000.0*1./12.
G=12000000.0
TH=E/(CAY*G)
GTH=100.
X=PI/GTH
Y=3.*X
XY=5.*X
OM(1)=E*X**4/(RO+RO*TH*X**2)
C(1)=TH**2.*RO*OM(1)*X-TH*E*X**3.
C(2)=3.*TH*RO*OM(1)+TH**2.*RO*OM(1)*X**2.+E*X**2.-TH*E*X**4.
C(3)=-2.*E*X**3.
C(4)=0.0
C(5)=E*Y**4.-RO*OM(1)-TH*RO*OM(1)*Y**2.
Z(1)=4.*X**3.*C(1)-2.*X**2.*C(2)+X*C(3)+C(4)/2.
Z(2)=0.0
OM(2)=0.0
C(6)=0.0
C(8)=0.0
C(9)=E*XY**4.-RO*OM(1)-TH*RO*OM(1)*XY**2.
C(10)=0.0
C(11)=-2.*TH**2.*RO*OM(1)-TH*E*X**2.
C(12)= 9.*TH**2.*RO*OM(1)*X-6.*TH*E*X**3.
C(13)=TH**2.*RO*OM(1)*X**2.-TH*E*X**4.+7.*TH*RO*OM(1)
C(14)=-TH*E*X**2.-TH**2.*RO*OM(1)
C(15)=0.0
C(16)=TH*E*Y**4.-3.*TH*RO*OM(1)-TH**2.*RO*OM(1)*Y**2.-E*Y**2
C(17)=TH**2.*RO*OM(1)*Y-TH*E*Y**3.
C(18)=0.0
B(2)=0.0
B(3)=C(9)
S=C(5)/Z(1)
R=DABS(S)
B(4)=S*(C(10)-4.*X**2.*C(15))
B(5)=S*(2.*C(5)-4.*X**2.*C(16))
B(6)=S*(54.*X*C(3)+8.*X**3.*C(17))
B(7)=R**2*(-8.*X**3.*C(1)+8.*X**3.*C(12)-2.*X*C(3))
B(8)=R**2.*(16.*X**4.*C(11)+4.*X**2.*C(2))
B(9)=R**2.*(4.*X**2.*C(13)+16.*X**4.*C(14))
B(10)=0.0
ZETA(2)=.5*(2.*B(2)+B(4)+B(7)/2.-B(8)/2.+B(9)/2.-B(10))
ZETA(3)=.5*(2.*B(3)+B(5)-B(6)+B(7)/2.+B(8)/2.-B(9)/2.)
ZETA(4)=-.5*(B(4)-B(5)-B(6)-B(8)-B(9))
Z(3)=0.0
Z(4)=Z(1)
Z(6)=C(4)/2.+2.*X*C(3)-8.*C(2)*X**2.+32.*C(1)*X**3.
Z(5)=Z(6)+16.*C(2)*X**2
EN(1)=C(5)/Z(1)
EN(3)=ZETA(3)/Z(6)
```

```
      EN(2)=(ZETA(2)-EN(3)*Z(5))/Z(4)
      OM(3)=ZETA(4)/(RO*(1.+TH*X**2))
      C(7)=-RO*(OM(3)+TH*OM(3)*X**2)
      B(1)=C(7)
      ZETA(1)=.5*(2.*B(1)-B(4)+B(5)+B(6)+B(8)+B(9))
      WRITE(6,900)
      WRITE(6,301) OM(1),OM(2),OM(3)
      WRITE(6,901)
      WRITE(6,302) (C(M),M=1,18)
      WRITE(6,902)
      WRITE(6,302) (Z(N),N=1,6)
      WRITE(6,904)
      WRITE(6,304) (B(IJ),IJ=1,10)
      WRITE(6,905)
      WRITE(6,305) (ZETA(JK),JK=1,4)
      WRITE(6,906)
      WRITE(6,301) (EN(L),L=1,3)
  301 FORMAT(3D15.7,//)
  302 FORMAT(6D15.7,//)
  304 FORMAT(5D15.7,//)
  305 FORMAT(4D15.7,//)
  900 FORMAT(   ' OMEGA 1,2 AND 3',//)
  901 FORMAT(   ' K 1 THRU 18',//)
  902 FORMAT(   ' Z 1 THRU 6',//)
  904 FORMAT(   ' BETA 1 THRU 10',//)
  905 FORMAT(   ' ZETA 1 THRU 4',//)
  906 FORMAT(   ' M 1 THRU 3',//)
      DO 1 I=1,10
      U(1)=1.0
      A(I)=U(I)*PI/GTH
      BB(I)=(U(I)-2.)*PI/GTH
      XX(I)=(U(I)+2.)*PI/GTH
      YY(I)=(U(I)-4.)*PI/GTH
      ZZ(I)=(U(I)+4.)*PI/GTH
      U(I+1)=U(I)+1.0
    1 CONTINUE
      DO 2 J=1,10
      O(J)=(E*A(J)**4.)/(RO*(1.+TH*A(J)**2.))
    2 CONTINUE
C
C     LOOP FOR COMPUTING THE VARIABLE AREA
C
      T=2.*PI/GTH
      V=4.*PI/GTH
      DO 6 IK=1,101
      XYZ(1)=0.0
      XYZ(IK+1)=XYZ(IK)+1.0
    6 CONTINUE
```

```
      DO 101 K=1,5
      EPS(1)=.01
      DO 100 I=1,101
      W(1)=0.0
      H(1)=EN(1)*DCOS(T* W(I))
      H(2)=EN(2)*DCOS(T* W(I))+EN(3)*DCOS(V* W(I))
      AT(I)=EPS(K)*H(1)+EPS(K)**2*H(2)
      W(I+1)=W(I)+1.0
  100 CONTINUE
      WRITE(6,907) K
      WRITE(6,300) (AT(J), J=1,100)
      WRITE(6,306)
  907 FORMAT(  ' A=EPS*A(1)+EPS**2*A(2) FOR EPS=.0',I1,//)
  300 FORMAT(5D15.7,////)
  306 FORMAT(  '**************************************************',//)
      EPS(K+1)=EPS(K)+.01
      DO 10 N=1,101
      TA(N)=AT(N)
   10 CONTINUE
      CALL DRAW(101,TA,XYZ,0,0,LABEL,ITITLE,2.0,20.0,0,1,2,2,2,5 ,1,LAST
     1)
  101 CONTINUE
C
C     LOOP FOR EIGEN FREQUENCIES AND EIGENVALUES
C
      BO=1.0
      Q(1)=E
      DO  7 K=1,10
      Q(2)=TH*RO*O(K)
      Q(3)=RO*O(K)
      Q(5)=BO*T**3.*EN(1)*(TH**2.*RO*O(K)*A(K)-TH*E*A(K)**3.)-BO*T*EN(1)
     1*(2.*E*A(K)**3.)
      Q(6)=BO*EN(1)*(-2.*E*A(K)**4.+2.*RO*O(K)+2.*TH*RO*O(K)*A(K)**2.)+B
     1O*T**2.*EN(1)*(3.*TH*RO*O(K)+TH**2.*RO*O(K)*A(K)**2.+E*A(K)**2.-TH
     2*E*A(K)**4.)
      Q(7)=0.0
      Q(8)=2.*E*BO*A(K)**3
      Q(9)=BO*(3.*TH*RO*O(K)-TH*E*A(K)**4.+TH**2.*RO*O(K)*A(K)**2.+E*A(K
     1)**2.)
      Q(10)=BO*(TH*E*A(K)**3.-TH**2.*RO*O(K)*A(K))
      OO(K)=0.0
      BN(K)=-(Q(6)+Q(5))/(2.*(E*BB(K)**4-TH*RO*O(K)*BB(K)**2-RO*O(K)))
      DN(K)=(Q(5)-Q(6))/(2.*(E*XX(K)**4 -TH*RO*O(K)*XX(K)**2 -RO*O(K)))
      BN(1)=0.0
      Q(4)=BO*(RO+TH*RO*A(K)**2)*OO(1)
      Q(12)=0.
      Q(13)=DN(K)*(RO+RO*TH*BB(K)**2)*OO(K)
      Q(14)=BN(K)*(E*BB(K)**4-RO*O(K)-TH*RO*O(K)*BB(K)**2)
```

```
      IF(K.EQ.1)Q(14)=0.0
      Q(15)=DN(K)*(E*XX(K)**4-RO*O(K)-TH*RO*O(K)*XX(K)**2)
      Q(16)=0.0
      Q(17)=BN(K)*2.*E*BB(K)**3
      Q(18)=DN(K)*2.*E*XX(K)**3
      Q(19)=Q(8)
      Q(20)=BO*(2.*TH**2*RO*O(K)+TH*E*A(K)**2)
      Q(21)=BO*(9.*TH**2*RO*O(K)*A(K)-6.*TH*E*A(K)**3)
      Q(22)=BO*(-TH*E*A(K)**4+TH**2*RO*O(K)*A(K)**2+7.*TH*RO*O(K))
      Q(23)=BO*(TH*E*A(K)**2+TH**2*RO*O(K))
      Q(24)=BN(K)*(TH*E*BB(K)**4-3.*TH*RO*O(K)-TH**2*RO*O(K)*BB(K)**2-E*
     1BB(K)**2)
      Q(25)=DN(K)*(TH*E*XX(K)**4-3.*TH*RO*O(K)-TH**2*RO*O(K)*XX(K)**2-E*
     1XX(K)**2)
      Q(26)=BC*(3.*TH*RO*OO(K)+TH**2*RO*OC(K)*A(K)**2)
      Q(27)=-Q(9)
      Q(28)=BN(K)*(-TH*E*BB(K)**3+TH**2*RO*O(K)*BB(K))
      Q(29)=DN(K)*(-TH*E*XX(K)**3+TH**2*RO*O(K)*XX(K))
      Q(30)=BO*TH**2*RO*OC(K)*A(K)
      Q(31)=-Q(10)
      P(1)=-Q(9)*EN(2)*T**2
      P(2)=-Q(9)*EN(3)*V**2
      P(3)=-EN(2)*Q(8)*T-EN(2)*Q(10)*T**3
      P(4)=EN(3)*Q(8)*V+Q(10)*EN(3)*V**3
      P(6)=Q(12)
      P(7)=Q(13)
      P(8)=2.*EN(1)*Q(14)-EN(1)*T**2*Q(24)
      P(9)=2.*EN(1)*Q(15)-EN(1)*T**2*Q(25)
      P(10)=-EN(1)*T*Q(17)+EN(1)*T**3*Q(28)
      P(11)=EN(1)*T**3*Q(29)-EN(1)*T*Q(18)
      P(12)=EN(1)**2*T**3*Q(31)-EN(1)**2*T*Q(19)-EN(1)**2*T**3*Q(21)
      P(13)=EN(1)**2*T**4*Q(20)+EN(1)**2*T**2*Q(27)
      P(14)=-EN(1)**2*T**2*Q(22)+EN(1)**2*T**4*Q(23)
      DEL(1)=.5*(P(1)+P(3)+2.*P(7))
      DEL(2)=.5*(P(1)-P(3)+2.*P(6))
      DEL(3)=.5*(P(2)-P(4)-P(9)    +P(11)+P(12)/2.+P(13)/2.-P(14)/2.)
      DEL(4)=.5*(P(2)+P(4)-P(8)-P(10)-P(12)/2.+P(13)/2.-P(14)/2.)
      DEL(6)=.5*(        -P(8)-P(9)    +P(10)-P(11)+P(13)+P(14))
      OOO(K)=-DEL(6)/(BO*RO*(1.+TH*    A(K)**2))
      IF(K.EQ.1) OOO(K)= (DEL(2)-DEL(6))/(BO*RO*(1.+TH*A(K)**2))
      IF(K.EQ.2) OOO(K)= (DEL(4)-DEL(6))/(RO*(1.+TH*T**2))
      Q(11)=BO*(RO+TH*RO*A(K)**2)*OOO(K)
      P(5)=Q(11)
      DEL(5)=.5*(2.*P(5)-P(8)-P(9)    +P(10)-P(11)+P(13)+P(14))
      BT(K)=DEL(1)/(E*XX(K)**4-RO*O(K)-TH*RO*O(K)*XX(K)**2)
      IF(K.EQ.1) BT(K)=0.0
      DT(K)=DEL(2)/(E*BB(K)**4-RO*O(K)-TH*RO*O(K)*BB(K)**2)
      IF(K.EQ.1) DT(K)=0.0
```

```
      IF(K.EQ.2) DT(K)=0.0
      ET(K)=DEL(3)/(E*ZZ(K)**4-RO*O(K)-TH*RO*O(K)*ZZ(K)**2)
      FT(K)=DEL(4)/(E*YY(K)**4-RO*O(K)-TH*RO*O(K)*YY(K)**2)
      IF(K.EQ.1) FT(K)=0.0
      IF(K.EQ.2) FT(K)=0.0
      IF(K.EQ.4) FT(K)=0.0
      WRITE(6,26)
      WRITE(6,20)
      WRITE(6,26)
      WRITE(6,222)  O(K),OO(K),OOO(K)
      WRITE(6,26)
      WRITE(6,21)
      WRITE(6,26)
      WRITE(6,333)(Q(M),M=1,31)
      WRITE(6,26)
      WRITE(6,22)
      WRITE(6,26)
      WRITE(6,333)  (P(I),I=1,14)
      WRITE(6,26)
      WRITE(6,23)
      WRITE(6,26)
      WRITE(6,333)   (DEL(J),J=1,6)
      WRITE(6,26)
      WRITE(6,24)
      WRITE(6,26)
      WRITE(6,999) BN(K),DN(K)
      WRITE(6,26)
      WRITE(6,25)
      WRITE(6,26)
      WRITE(6,701) BT(K),DT(K),ET(K),FT(K)
  222 FORMAT(3D15.7,//)
  333 FORMAT(5D15.7,//)
  701 FORMAT(4D15.7,//)
  999 FORMAT(2D15.7,//)
   20 FORMAT(   ' OMEGA 1,2 AND 3',//)
   21 FORMAT(   ' Q 1 THRU 31',//)
   22 FORMAT(   ' PHI 1 THRU 14',//)
   23 FORMAT(   ' DELTA 1 THRU 6',//)
   24 FORMAT(   ' AMPLITUDES FOR FIRST CRDER',//)
   25 FORMAT(   ' AMPLITUDES FOR SECOND CRDER',//)
      ESP=.05
      DO 8 IK=1,101
      DEF(1)=BN(K)*DSIN(BB(K)*W(IK))+DN(K)*DSIN(XX(K)*W(IK))
      DEF(2)=BT(K)*DSIN(XX(K)*W(IK))+DT(K)*DSIN(BB(K)*W(IK))+ET(K)*DSIN(
     1ZZ(K)*W(IK))+FT(K)*DSIN(YY(K)*W(IK))
      DEFT(IK)=DSIN(A(K)*W(IK))+ESP*DEF(1)+ESP**2*DEF(2)
    8 CONTINUE
      DO 9 IJ=1,101
```

```
      WS(IJ)=W(IJ)
      DEFTS(IJ)=DEFT(IJ)
    o CONTINUE
      WRITE(6,26)
      WRITE(6,99) K
   99 FORMAT(  ' MODE SHAPE NUMBER',1X,I2,//)
      WRITE(6,26)
      WRITE(6,444) (DEFT(MN),MN=1,101)
  444 FORMAT(5D15.7,//)
   26 FORMAT(  ' ************************************************',//)
      CALL DRAW(101,DEFTS,WS,0,0,LABEL,ITITLE,2.0,20.0,0,1,2,2,2,5 ,1,LA
     1ST)
    7 CONTINUE
C
C     LOOP FOR COMPUTING EIGEN FREQUENCIES
C
      DO 700 I=1,10
      OX(I)=O(I)+ESP**2*OOO(I)
      OMT(I)=DSQRT(OX(I))
  700 CONTINUE
      WRITE(6,603)
  603 FORMAT(//)
      WRITE(6,26)
      WRITE(6,602)
      WRITE(6,26)
      WRITE(6,601) (OMT(N),N=1,10)
  602 FORMAT(  ' THE FIRST TEN EIGEN FREQUENCIES',//)
  601 FORMAT(5D15.7)
      STOP
      END
 T. D. PESTORIUS PES20507
```

APPENDIX D

INTERMEDIATE FORMS OF EQUATIONS

Referring to page 25 the differential equation for $A_2(x)$ is:

$$
\begin{aligned}
&A_0\left[A_2''' K_1 \cos\alpha + A_2'' K_2 \sin\alpha - A_2' K_3 \cos\alpha\right] = A_0^2 \{EC_1 y_2^{IV} - \rho\omega_2^2 y_0 \\
&- \rho\omega_0^2 y_2 - \rho\omega_1^2 y_1 + \Theta\rho\omega_2^2 y_0'' + \Theta\rho\omega_0^2 y_2'' + \Theta\rho\omega_1^2 y_1''\} + 2A_0A_1\{ \\
&EC_1 y_1^{IV} - \rho\omega_1^2 y_0 - \rho\omega_0^2 y_1 + \Theta\rho\omega_1^2 y_0'' + \Theta\rho\omega_0^2 y_1''\} + A_0A_1' \\
&\{2EC_1 y_1'''\} + A_1A_1'\{2EC_1 y_0'''\} + A_1''^2\{-2\Theta^2\rho\omega_0^2 y_0 + \Theta EC_1 y_0''\} + \\
&A_1'A_1''\{9\Theta^2\rho\omega_0^2 y_0' + 6\Theta EC_1 y_0'''\} + A_1'^2\{-\Theta^2\rho\omega_0^2 y_0'' - \Theta EC_1 y_0^{IV} \\
&+ 7\Theta\rho\omega_0^2 y_0\} - A_1'A_1'''\{\Theta EC_1 y_0'' - \Theta^2\rho\omega_0^2 y_0\} + A_0A_1''\{\Theta EC_1 y_1^{IV} \\
&- 3\Theta\rho\omega_0^2 y_1 - 3\Theta\rho\omega_1^2 y_0 + \Theta^2\rho\omega_0^2 y_1'' + \Theta^2\rho\omega_1^2 y_0'' + EC_1 y_1''\} \\
&+ A_1A_1''\{\Theta EC_1 y_0^{IV} - 3\Theta\rho\omega_0^2 y_0 + \Theta^2\rho\omega_0^2 y_0'' + EC_1 y_0''\} - A_0A_1''' \\
&\{\Theta EC_1 y_1''' + \Theta^2\rho\omega_0^2 y_1' + \Theta^2\rho\omega_1^2 y_0'\} - A_1A_1'''\{\Theta EC_1 y_0''' + \Theta^2\rho\omega_0^2 y_0'\}
\end{aligned}
$$

Substitution of $A_1(x)$, y_0, y_1, and y_2, and subsequent reduction gives:

$$
\begin{aligned}
&A_0\left[A_2''' K_1 \cos\alpha + A_2'' K_2 \sin\alpha - A_2' K_3 \cos\alpha\right] = \beta_1 \sin\alpha + \beta_2 \sin 3\alpha \\
&+ \beta_3 \sin 5\alpha + \beta_4 \sin\alpha \cos 2\alpha + \beta_5 \sin 3\alpha \cos 2\alpha - \\
&\beta_6 \sin 2\alpha \cos 3\alpha + \beta_7 \cos\alpha \cos 2\alpha \sin 2\alpha + \\
&\beta_8 \sin\alpha \cos^2 2\alpha + \beta_9 \sin\alpha \sin^2 2\alpha - \beta_{10} \cos\alpha \sin 2\alpha
\end{aligned}
$$

After further reduction, using trigonometric identities, equation (31) is produced.

The basic differential equation for y_{n2} is similar to that for $A_2(x)$. The only difference is that y_{n2} is treated as the unknown. After substitution of A_1, A_2, y_{n0}, and y_{n1}, the equation takes the form:

$$
\begin{aligned}
A_0^2\left[Q_1 y_{n2}^{IV} + Q_2 y_{n2}'' - Q_3 y_{n2}\right] = {} & \phi_1 \cos 2\alpha \sin n\alpha + \phi_2 \cos 4\alpha \sin n\alpha \\
& + \phi_3 \sin 2\alpha \cos n\alpha - \phi_4 \sin 4\alpha \cos n\alpha + \phi_5 \sin n\alpha + \phi_6 \sin px \\
& + \phi_7 \sin qx - \phi_8 \cos 2\alpha \sin px - \phi_9 \cos 2\alpha \sin qx \\
& + \phi_{10} \sin 2\alpha \cos px + \phi_{11} \sin 2\alpha \cos qx + \phi_{12} \cos 2\alpha \sin 2\alpha \cos n\alpha \\
& + \phi_{13} \cos^2 2\alpha \sin n\alpha + \phi_{14} \sin^2 2\alpha \sin n\alpha
\end{aligned}
$$

Application of trigonometric identities to this equation produces equation (46).

BIBLIOGRAPHY

1. Ames, W. F. and Sontowski, J. F., "Multiparameter Perturbation Solution of Algebraic Equations," Journal of Applied Mechanics, ASME, March 1966, pp. 218-219.

2. Anderson, R. A., "Flexural Vibrations in Uniform Beams According to the Timoshenko Theory," Journal of Applied Mechanics, ASME, December 1953, pp. 504-510.

3. Bellman, R., Perturbation Techniques in Mathematics, Physics, and Engineering, Holt, Rinehart and Winston, Inc., 1964.

4. Cowper, G. R., "The Shear Coefficient in Timoshenko's Beam Theory," Journal of Applied Mechanics, ASME, June 1966, pp. 335-339.

5. Fung, Y. C., Foundations of Solid Mechanics, Prentice-Hall, Inc., 1966.

6. Huang, T. C., "The Effect of Rotary Inertia and of Shear Deformation on the Frequency and Normal Mode Equations of Uniform Beams with Simple end Conditions," Journal of Applied Mechanics, ASME, December 1961, pp. 579-584.

7. Irving, J. and Mullineux, N., Mathematics in Physics and Engineering, Academic Press, New York and London, 1959.

8. Jacobsen, L. S. and Ayre, R. S., Engineering Vibrations, McGraw-Hill Book Company, Inc., 1958.

9. Mindlin, R. D. and Deresiewicz, H., Timoshenko's Shear Coefficient for Flexural Vibrations of Beams, Columbia University, Department of Civil Engineering, June 1953.

10. Timoshenko, S. P., "On the Correction for Shear of the Differential Equations for Transverse Vibrations of Prismatic Bars," Philosophical Magazine, Vol. 41, 1921, pp. 744-746.

11. Timoshenko, S. P., "On the Transverse Vibrations of Bars of Uniform Cross Section," Philosophical Magazine, Vol. 43, 1922, pp. 125-131.

12. Timoshenko, S. P., Vibration Problems in Engineering, McGraw-Hill Book Company, Inc., 1955.

13. Whittaker, E. T. and Watson, G. N., A Course of Modern Analysis, Cambridge University Press, 1965.

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Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Synthesis of Doubly Symmetric Vibrating Beams Including Shear

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Master's Thesis; June 1969

5. AUTHOR(S) (First name, middle initial, last name)

Thomas Daniel Pestorius

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
June 1969	54	13

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Naval Postgraduate School Monterey, California 93940

13. ABSTRACT

A fundamental, non-linear, ordinary differential equation is derived for the flexural vibrations of a non-uniform beam, including shear.

This equation is solved with the specializing assumptions of a constant shear coefficient and that the moment of inertia is directly proportional to the cross sectional area.

A perturbation method is used and the eigen frequencies and mode shapes obtained are functions of a perturbation parameter ϵ. These solutions are accurate up to the second order of ϵ.

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Nonuniform Beams						
Beams						
Timoshenko Beams						
Vibration of Beams						
Beam Optimization						

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S/N 0101-807-6821

Security Classification

A-31409